arXiv:1007.2833v3 [math.AP] 9 Dec 2010

# Pathwise Solutions of the 2-D Stochastic Primitive Equations

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Dedicated to Alain Bensoussan on the occasion of his 70th birthday.

**Abstract**

In this work we consider a stochastic version of the Primitive Equations (PEs) of the ocean and the atmosphere and establish the existence and uniqueness of pathwise, strong solutions. The analysis employs novel techniques in contrast to previous works [18], [23] in order to handle a general class of nonlinear noise structures and to allow for physically relevant boundary conditions. The proof relies on Cauchy estimates, stopping time arguments and anisotropic estimates.

## 1 Introduction

The Primitive Equations (PEs) are widely regarded as a fundamental description of geophysical scale fluid flows. They provide the analytical core of large General Circulation Models (GCMs) that are at the forefront of numerical simulations of the earth's ocean and atmosphere (see e.g. [47]). In view of the wide progress made in computation the need has appeared to better understand and model some of the uncertainties which are contained in these GCMs. This is the so called problem of "parameterization". Besides all of the physical forms of parameterization [47, 41, 40] , stochastic modeling has appeared as one of the major modes in the contemporary evolution of the field (see [17, 36, 37, 43, 28, 32, 7, 50] and also [22]). In this context there is a clear need to better understand the numerical and analytical underpinnings of stochastic partial differential equations.

In the present article we will establish the global well-posedness of the stochastically forced Primitive Equations of the ocean in dimension two. While this system has been treated in a simplified form in previous works, for the case of additive noise [18] and nonphysical boundary conditions [23], our aim here is to go further and treat a more physically realistic version of these equations

in the context of a multiplicative noise. In the formulation herein we face two new fundamental difficulties in contrast to previous work. Firstly, due to the imposed boundary conditions we lose higher order cancelations in the nonlinear terms. This complicates the a priori estimates which in turn prevent the usage of more direct compactness arguments adopted in, [8], [23]. On the other hand, due to the nonlinear multiplicative noise structure, the system may not be transformed into a random PDE as in [18]. For this reason we are not able to treat the probabilistic dependence as a parameter in the problem. The analysis therefore requires the usage of advanced tools both from stochastic analysis, namely continuous time martingale theory and stopping time arguments, and PDE theory which we treat in detail in a separate work [21].

A significant literature exists concerning the Navier-Stokes equations driven by a multiplicative volumic white noise forcing. See [6, 48, 13, 12, 19, 34, 16, 8, 4, 10, 33, 20]. While our point of view is similar to some of these works we would like to point out that the Primitive Equations, not withstanding very recent results on global well-posedness in 3D, are technically more involved than the Navier- Stokes equations.

This article is dedicated to Alain Bensoussan on the occasion of his 70th birthday with friendship and admiration, and, for the second author (RT), sweet reminiscences of many interactions, from Junior High School, to the early papers on stochastic partial differential equations [5], [6], on the subject of this article, and to many more interactions over the years.

## 1.1 Presentation of the 2D Stochastic PEs

The 2D stochastic Primitive Equations take the form

$$\partial_t u + u\partial_x u + w\partial_z u - \nu\Delta u - fv + \partial_x p = F_u + \sigma_u(\mathbf{v}, T)\dot{W}_1, \tag{1.1a}$$

$$\partial_t v + u\partial_x v + w\partial_z v - \nu\Delta v + fu = F_v + \sigma_v(\mathbf{v}, T)\dot{W}_2, \tag{1.1b}$$

$$\partial_z p = -\rho g, \tag{1.1c}$$

$$\partial_x u + \partial_z w = 0, \tag{1.1d}$$

$$\partial_t T + u\partial_x T + w\partial_z T - \mu\Delta T = F_T + \sigma_T(\mathbf{v}, T)\dot{W}_3, \tag{1.1e}$$

$$\rho = \rho_0(1 - \beta_T(T - T_0)). \tag{1.1f}$$

This two dimensional model may be derived from the classical three dimensional formulation by positing invariance in one of the horizontal directions, namely the $y-$ (south-north) direction. Here $(\mathbf{v}, w) = (u, v, w)$, $T$, $\rho$ denote respectively the flow field, the temperature and the density of the fluid being modeled. The coefficients $\nu$, $\mu$ account for the molecular viscosity and the rate of heat diffusion. A further parameter $f$, which is a function of the earth's rotation, appears in an antisymmetric term and is taken constant (see below). The terms $F_u$, $F_v$ and $F_T$ correspond to external sources of horizontal momentum and heat. While the first two terms do not usually appear in practice we retain them here for mathematical generality and to allow for the possible treatment, not carried out here, of non-homogenous boundary conditions.

The white noise processes $\dot{W}_i$, the raison d'être of the present work may be written in the expansions

$$\begin{pmatrix} \sigma_u(\mathbf{v},T)\dot{W}_1 \\ \sigma_v(\mathbf{v},T)\dot{W}_2 \\ \sigma_T(\mathbf{v},T)\dot{W}_3 \end{pmatrix} = \sigma_{\mathbf{v},T}(U)\dot{W} = \sum_k \sigma_{\mathbf{v},T}^k(U)\dot{W}^k. \tag{1.2}$$

The $\dot{W}^k$s may be interpreted as the time derivatives of a sequence of independent standard 3-D brownian motions. However, since the sample paths of brownian motion are nowhere differentiable we make rigorous sense of (1.1a), (1.1b) and (1.1e) in a time integrated sense, appealing to the theory of stochastic integration which we consider in the Itō sense. From the physical point of view these terms may be introduced in the model as a means to "parameterize" physical and numerical uncertainties.

We consider the evolution of (1.1) over a rectangular domain $\mathcal{M} = (0,L) \times (-h,0)$ and label the boundary $\Gamma_i = (0,L) \times \{0\}$, $\Gamma_b = (0,L) \times \{-h\}$ and $\Gamma_l = \{0,L\} \times (-h,0)$. We posit the physically realistic boundary conditions

$$\partial_z \mathbf{v} + \alpha_{\mathbf{v}} \mathbf{v} = 0, \quad w = 0, \quad \partial_z T + \alpha_T T = 0, \quad \text{ on } \Gamma_i, \tag{1.3a}$$
$$\mathbf{v} = 0, \quad \partial_x T = 0, \quad \text{ on } \Gamma_l, \tag{1.3b}$$
$$\mathbf{v} = 0, \quad w = 0, \quad \partial_z T = 0, \quad \text{ on } \Gamma_b.\text{[1]} \tag{1.3c}$$

The equations and boundary conditions (1.1), (1.3) are supplemented by initial conditions for $u$, $v$ and $T$, that is

$$u = u_0, \quad v = v_0, \quad T = T_0, \quad \text{ at } t = 0. \tag{1.4}$$

The Primitive equations may be derived from the compressible Navier-Stokes equations with a combination of empirical observation and scale analysis. In particular, since deviations of the density of the fluid from a mean value are small at geophysical scales, the so—called Boussinesq approximation justifies treating the flow as incompressible.[2] Another crucial feature, that the ocean and atmosphere form a thin later on the earth surface leads to the hydrostatic approximation which reduces the third momentum equation to (1.1c). Beyond its obvious numerical significance, this anisotropy in the governing equations has many interesting theoretical consequences. We refer the interested reader to the classical texts [14] and [35] for an introduction from the physical point of view.

Particularly in view of the numerous complications involved in extending the existing deterministic model to the stochastic setting we have made some

[1]Many of these boundary conditions may be non-homogenous (that is that they may include suitable forcing) in general. We will consider only the homogeneous case here.

[2]The Boussinesq approximation concerns the oceans. For the atmosphere we arrive at very similar equations by considering the pressure as the vertical coordinate, but, for the sake of simplicity, the emphasis here will be on the case of the oceans.

simplifications for the purposes of clarity of presentation. The equation (1.1) is a description of the earth's ocean but all of what follows can be easily extended to the PEs of the atmosphere or of the coupled atmosphere-ocean system (see [29]). We assume moreover that the $\beta$-plane approximation is valid. This assumption, that the earth is locally flat, is appropriate for regional climatological studies. Of course, for larger scales one must include additional terms that account for the curvature of the earth. Since it is convenient to work in the rotating reference frame of the earth's surface, an additional antisymmetric term appears in the momentum equations. The Coriolis parameter in this term, which we denote by $f$, depends on the earth's angular velocity and the local latitude of the region under investigation. In the context of the $\beta$-plane approximation, $f$ is usually a linear function of $y$, $f = f_0(1 + \beta y)$. Here we take $f$ to be constant, but once again the proof is easily modified to treat the more general case.

Several other terms have been simplified or deleted which may be reintroduced in their full form with no new complications to the mathematical framework or to the proof of the main theorem. We neglect the density dependence on the salinity of the ocean. We therefore drop the diffusion equation that accounts for variations in salt concentration in the fluid. We also ignore further, possibly anisotropic, diffusion terms that may appear in both the momentum and temperature equations to account for subgrid scale processes, the so called eddy diffusion terms. Finally, as noted above, we consider only the case of homogenous boundary conditions.

Dating back to a series of seminal works in the early 90's [31], [30], and [29] a significant mathematical literature has developed around the Primitive Equation. In a significant breakthrough, the global well posedness in 3-D was established [11], [25], [26]. Subsequent work of [27] developed alternative proofs, which allow for the treatment of physically relevant boundary conditions. For the two dimensional deterministic setting we mention [38], [9] where both the cases of weak and strong solutions are considered. Despite these breakthroughs in the 3-D system, the 2-D primitive equations seem to be significantly more difficult mathematically than the 2-D Navier-Stokes equations. For instance, it is still an open problem as to whether weak solutions of the Primitive equations in the deterministic setting are unique. This is a classical exercise for the 2-D Navier Stokes equations. In any case we refer the interested reader to the recent survey papers [44] and [39] (appearing in [1]) which provide a systematic overview of deterministic theory. Note that, in regards to notational conventions and earlier deterministic results the present article relies heavily on this later work.

While the deterministic mathematical theory is now on a firm ground the stochastic theory remain underdeveloped. In [23] the existence of pathwise, $z$-weak solutions was established for a simplified model with nonlinear multiplicative noise and non-physical boundary conditions. A more extended system was considered in [18] again for the so–called $z$-weak solutions but with additive noise and periodic boundary conditions. Adapting the methods of [11] the 3-d case with additive noise and nonphysical boundary conditions was recently treated in [2].

In contrast, beginning with the seminal work [6], extensive investigations for the stochastic Navier-Stokes equations have been undertaken. For weak or martingale solutions we mention [48], [13], [12], [19], [34] and further references therein. Regarding pathwise solutions we mention [16], [8], [4], [10], [33]. In recent joint work of the first coauthor [20] the local and global theory of pathwise solution in $H^1 = W^{1,2}$ was established. Some of the tools and techniques developed in this final reference play a central role herein.

In the present work we will establish the global existence and uniqueness of a pathwise solution to (1.1), supplemented by (1.3) for all $U_0 = (u, v, T) \in (H^1)^3$. We conclude this introduction with an outline of the basic difficulties we encounter along with the main steps in the proof.

## 1.2 Basic Estimates and some Difficulties Particular to the Stochastic Case

The first step in the proof is to establish the local existence, up to a strictly positive stopping time $\tau$, of a solution $U$ for (1.1) in $L_t^\infty H_x^1 \cap L_t^2 H_x^2$. Here and throughout the rest of the work $U$ stands for the (prognostic) unknowns in the problem, $U = (u, v, T) = (\mathbf{v}, T)$; $U^{(n)}$ will denote some Galerkin approximation of $U$. Having implemented a Galerkin scheme the passage to the limit is delicate as it is not evident a priori how to uniformly choose $\tau > 0$ such that

$$\sup_n \mathbb{E}\left(\sup_{0\le t\le\tau} |U^{(n)}|_{H^1}^2 + \int_0^\tau |U^{(n)}|_{H^2}^2\right) < \infty.$$

Even if such a $\tau$ were to be found it would remain unclear how to infer the necessary sub-sequential (strong) compactness without changing the underlying stochastic basis. To overcome these difficulties we follow [20] and perform Cauchy type estimates for the Galerkin solutions $\{U^{(n)}\}_{n\ge 1}$ associated with (1.1) up to a carefully chosen sequence of stopping times. Since we have sufficient uniform control of the growth of $U^{(n)}$ at time zero we are able to pass to the limit almost surely up to a strictly positive time. Note that this stage of the investigation required us to establish some novel bounds on the nonlinear portion of the equation in $H^1$ (see (2.15) below) and to make careful use of the equivalence of some fractional order spaces. Since a significant portion of this analysis is non-probabilistic in character, we have separated these delicate and technical points to a separate work, [21].

With a local solution $(U, \tau) = ((u, v, T), \tau)$ in hand, further a posteriori estimates are needed to preclude the possibility of a finite time blowup. In previous work in the deterministic setting (which corresponds to the admissible case, $\sigma \equiv 0$) successive estimates on $U$, $\partial_z u$ and $\partial_x u$ in $L_t^\infty L^2 \cap L_t^2 H^1$ were conducted to finally obtain an estimate for $U$ in $L_t^\infty H_x^1 \cap L_t^2 H_x^2$. See [39]. For the present stochastic setting several difficulties emerge which prevent a trivial repetition of these estimates.

The first difficulty appears when one tries to make estimates for $\partial_z u$. If, on the one hand, we take $\partial_z$ of (1.1a) and then apply Itō's formula to determine

an evolution equation for $|\partial_z u|^2_{L^2(\mathcal{M})}$, we encounter terms of the form

$$\int_{\mathcal{M}} \partial_{zzz} u \partial_z u d\mathcal{M}.$$

Due to (1.3a) and (1.3c) second order terms occur on the boundary that seem to be intractable a priori. If, on the other hand, following [39, Section 3.3.4], we attempt to multiply (1.1a) by $Q(-\partial_{zz} u)$ it is not clear what the appropriate stochastic interpretation of $du \cdot Q(-\partial_{zz} u)$ should be. Here $Q$ is the orthogonal complement of the vertical averaging operator and is needed to get rid of the pressure in the governing equations (cf. (4.12) and the remarks immediately following).

To address these difficulties we introduce an auxiliary linear stochastic evolution system with a diffusion governed by the now established local solution of the original system. We use this system to "subtract off" the noise terms from (1.1a) at the cost of a number of new random terms which we must estimate. While we are indeed able to treat these terms, at each order our estimates require almost sure bounds (in $\omega$) on the norms of the solution at the previous order. For this reason an involved stopping time argument must be employed at the final step. Here we make repeated use of a novel abstract result concerning a generic class of stochastic processes (see Proposition 5.1) which streamlines the analysis.

# 2 Abstract Setting

We begin with a review of the mathematical setting for the stochastic Primitive Equations and define the pathwise solutions we will consider in this work. The deterministic and stochastic preliminaries are treated successively. For the deterministic elements we largely follow [39], to which we refer the reader for a more detailed treatment. For more theoretical background on the general theory of stochastic evolution systems we mention the classical book [15] or the more recent treatment in [42].

## 2.1 The Hydrostatic Approximation

The hydrostatic approximation, in concert with the incompressibility and the boundary conditions leads one to several simple observations that allow a useful reformulation of (1.1). This will motivate the mathematical set-up below.

First we consider the third component of the flow $w$. Notice that by integrating (1.1d) and making use of the boundary condition (1.3a) for $w$ we infer that

$$w(x, z) = -\int_z^0 \partial_z w(x, \bar{z}) d\bar{z} = \int_z^0 \partial_x u(x, \bar{z}) d\bar{z}. \tag{2.1}$$

Accordingly $w = w(u)$ is seen to be an explicit functional of $u$[3]. Also notice that according to the boundary conditions (1.3a), (1.3c) we impose on $w$,

[3]Indeed, $w$, $p$ and $\rho$ are called *diagnostic* variables in geophysical fluid mechanics. By

$\int_{-h}^{0} \partial_x u d\bar{z} = 0$. This implies that $\int_{-h}^{0} u d\bar{z}$ is constant in $x$ and so, due to the lateral boundary condition (1.3b), we conclude that

$$\int_{-h}^{0} u dz = 0. \tag{2.2}$$

Next we consider the pressure. By integrating the hydrostatic balance equation (1.1c) and making use of the linear dependence of the density on the temperature (1.1f) we deduce

$$p_s(x) - p(x,z) = \int_z^0 \partial_z p(x,\bar{z}) d\bar{z} = -g\rho_0 \int_z^0 (1 - \beta_T(T(x,\bar{z}) - T_0)) d\bar{z}. \tag{2.3}$$

Here $p_s$ is the surface pressure, which is unknown and a function of the horizontal variable only. We have therefore decomposed the pressure into two components, the second of which couples the first momentum equation to the heat diffusion equation. Rearranging above and taking a partial derivative in $x$ we arrive at

$$\partial_x p = \partial_x p_s - \beta_T g \rho_0 \int_z^0 \partial_x T d\bar{z}. \tag{2.4}$$

With the above considerations we now rewrite (1.1) as:

$$\begin{aligned} \partial_t u + u\partial_x u + w(u)\partial_z u - \nu\Delta u - fv + \partial_x p_s - \beta_T g\rho_0 \int_z^0 \partial_x T d\bar{z} \\ = F_u + \sigma_u(\mathbf{v},T)\dot{W}_1, \end{aligned} \tag{2.5a}$$

$$\partial_t v + u\partial_x v + w(u)\partial_z v - \nu\Delta v + fu = F_v + \sigma_v(\mathbf{v},T)\dot{W}_2, \tag{2.5b}$$

$$w(u) = \int_z^0 \partial_x u d\bar{z}, \quad \int_{-h}^0 u dz = 0, \tag{2.5c}$$

$$\partial_t T + u\partial_x T + w(u)\partial_z T - \mu\Delta T = F_T + \sigma_T(\mathbf{v},T)\dot{W}_3, \tag{2.5d}$$

## 2.2 Basic Function Spaces

The main function spaces used are defined as follows. Take:

$$H := \left\{ U = (u,v,T) \in L^2(\mathcal{M})^3 : \int_{-h}^0 u dz = 0 \right\}.$$

We equip $H$ with the inner product[4]

$$(U, U^\sharp) := \int_{\mathcal{M}} \mathbf{v}\cdot\mathbf{v}^\sharp d\mathcal{M} + \int_{\mathcal{M}} TT^\sharp d\mathcal{M}, \quad U = (\mathbf{v},T), U^\sharp = (\mathbf{v}^\sharp, T^\sharp).$$

opposition $u$, $v$ and $T$ are referred to as *prognostic* variables and are the unknowns in an initial value problem which we set up below.

[4]One sometimes also finds the more general definition $(U,U^\sharp) := \int_{\mathcal{M}} \mathbf{v}\cdot\mathbf{v}^\sharp d\mathcal{M} + \kappa\int_{\mathcal{M}} TT^\sharp d\mathcal{M}$ with $\kappa > 0$ fixed. This $\kappa$ is useful for the coherence of physical dimensions and for (mathematical) coercivity. Since this is not needed here we take $\kappa = 1$.

Here and below we shall make use of the vertical averaging operator $P\phi = \frac{1}{h}\int_{-h}^{0}\phi(\bar{z})d\bar{z}$ and its orthogonal complement $Q\phi = \phi - P\phi$. Note that the projection operator $\Pi : L^2(\mathcal{M})^3 \to H$ may be explicitly defined according to $U \mapsto (Qu, v, T)$. We also define

$$V := \left\{ U = (u, v, T) \in H^1(\mathcal{M})^3 : \int_{-h}^{0} u dz = 0, \mathbf{v} = 0 \text{ on } \Gamma_l \cup \Gamma_b \right\}.$$

Here we take the inner product $((\cdot,\cdot)) = \nu((\cdot,\cdot))_1 + \mu((\cdot,\cdot))_2$ where, for given $U = (\mathbf{v}, T), U^\sharp = (\mathbf{v}^\sharp, T^\sharp)$

$$\begin{aligned}
((U, U^\sharp))_1 &:= \int_{\mathcal{M}} \partial_x \mathbf{v} \cdot \partial_x \mathbf{v}^\sharp + \partial_z \mathbf{v} \cdot \partial_z \mathbf{v}^\sharp \, d\mathcal{M} + \alpha_{\mathbf{v}} \int_{\Gamma_i} \mathbf{v} \cdot \mathbf{v}^\sharp \, dx, \\
((U, U^\sharp))_2 &:= \int_{\mathcal{M}} \partial_x T \partial_x T^\sharp + \partial_z T \partial_z T^\sharp \, d\mathcal{M} + \alpha_T \int_{\Gamma_i} T T^\sharp \, dx.
\end{aligned}$$

Note that under these definitions a Poincaré type inequality $|U| \leq C\|U\|$ holds for all $U \in H^1(\mathcal{M})^3 \supset V$. Moreover the norms $\|\cdot\|_{H^1}$, $\|\cdot\|$ may be seen to be equivalent over all of $H^1(\mathcal{M})^3$.

Even if $U$ is very regular many of the main terms in the abstract formulation of (2.5) do not belong to $V$ (see (2.6),(2.9), (2.10)) As such, we shall also make use of some additional auxiliary spaces:

$$\begin{aligned}
\tilde{V} &:= \left\{ U = (u, v, T) \in H^1(\mathcal{M})^3 : \int_{-h}^{0} u dz = 0, \mathbf{v} = 0 \text{ on } \Gamma_l \right\}, \\
\mathcal{Z} &:= \left\{ U = (u, v, T) \in H^1(\mathcal{M})^3 : \mathbf{v} = 0 \text{ on } \Gamma_l \right\}.
\end{aligned}$$

As for $V$ we endow both spaces with the norm $\|\cdot\|$. One may verify that $\Pi : \mathcal{Z} \to \tilde{V}$ and is continuous on $H^1(\mathcal{M})^3$.

Finally we take $V_{(2)} = H^2(\mathcal{M})^3 \cap V$ and equip this space with the classical $H^2(\mathcal{M})$ norm which we denote by $|\cdot|_{(2)}$. Since a considerable portion of the work below will consist in making estimates for the first momentum equation (1.1) (or equivalently (2.5a)) we set for simplicity

$$|u|_{L^2(\mathcal{M})} := |u|, \quad |\nabla u|_{L^2(\mathcal{M})} := \|u\|, \quad |u|_{H^2(\mathcal{M})} := |u|_{(2)},$$

for $u \in L^2(\mathcal{M})$ or $H^1(\mathcal{M})$ or $H^2(\mathcal{M})$. Note that since we will always use a lower case $u$ (or as needed $u^\sharp$, $u^\flat$) for the first component of elements in the spaces $H, V, V_{(2)}$ the context will be clear.

## 2.3 The deterministic framework

The linear second order terms in the equation are captured in the Stokes-type operator $A$ which is understood as a bounded operator from $V$ to $V'$ via $\langle AU, U^\sharp \rangle = ((U, U^\sharp))$. The additional terms in the variational formulation of this portion of the equation capture the Robin boundary condition (1.3a). They may be formally derived by multiplying $-\nu\Delta u, -\nu\Delta v, -\mu\Delta T$ in (2.5a),

(2.5b), (2.5d) by test functions $u^\sharp, v^\sharp, T^\sharp$, integrating over $\mathcal{M}$ and integrating by parts. We shall make use of the subspace $D(A) \subset V_{(2)}$ given by

$$D(A) = \{U = (\mathbf{v}, T) \in V_{(2)} : \partial_z \mathbf{v} + \alpha_{\mathbf{v}} \mathbf{v} = 0, \partial_z T + \alpha_T T = 0 \text{ on } \Gamma_i, \\ \partial_x T = 0 \text{ on } \Gamma_l, \partial_z T = 0 \text{ on } \Gamma_b\}.$$

On this space we may extend $A$ to an unbounded operator by defining

$$AU = \begin{pmatrix} -\nu Q \Delta u \\ -\nu \Delta v \\ -\mu \Delta T \end{pmatrix}, \quad U \in D(A).$$

Since $A$ is self adjoint, with a compact inverse $A^{-1} : H \to D(A)$ we may apply the standard theory of compact, symmetric operators to guarantee the existence of an orthonormal basis $\{\Phi_k\}_{k \geq 0}$ for $H$ of eigenfunctions of $A$ with the associated eigenvalues $\{\lambda_k\}_{k \geq 0}$ forming an unbounded, increasing sequence. Note that by the regularity results in [49] or [46] we have $\Phi_k \in D(A) \subset V_{(2)}$. Define

$$H_n = span\{\Phi_1, \ldots, \Phi_n\}.$$

Take $P_n$ and $Q_n = I - P_n$ to be the projections from $H$ onto $H_n$ and its orthogonal complement respectively. For $m > n$ let $P_m^n = P_m - P_n$.

Note that in some previous works, the second component of the pressure (cf. (2.4) and [39, Section 2]), is included in the definition of the principal linear operator $A$. Since this breaks the symmetry of $A$ we relegate such terms to a separate, lower order operator $A_p$, which we define from $V'$ via $\langle A_p U, U^\sharp \rangle := \kappa g \rho_0 \int_{\mathcal{M}} \int_z^0 T d\bar{z} \partial_x u^\sharp d\mathcal{M}, \forall U^\sharp \in V$. Taking into account the boundary conditions for $u^\sharp$ on $\Gamma_\ell$ $(x = 0, L)$, this may be extended to a map $A_p : V \to H$ via

$$A_p U = \begin{pmatrix} -\beta_T g \rho_0 Q \left( \int_z^0 \partial_x T d\bar{z} \right) \\ 0 \\ 0 \end{pmatrix}. \tag{2.6}$$

If $U \in D(A)$, $A_p U \in \tilde{V}$ and we have that

$$|A_p U| \leq c \|U\|, \quad \|A_p U\| \leq c |U|_{(2)}. \tag{2.7}$$

We next capture the nonlinear portion of (1.1). Accordingly we *define* the diagnostic function $w$ by setting

$$w(U) = w(u) = \int_z^0 \partial_x u d\bar{z}, \quad U = (u, v, T) \in V. \tag{2.8}$$

For $U = (\mathbf{v}, T), U^\sharp = (\mathbf{v}^\sharp, T^\sharp) \in V$ we take $B(U, U^\sharp) = B_1(U, U^\sharp) + B_2(U, U^\sharp)$ where

$$B_1(U, U^\sharp) := \begin{pmatrix} Q(u \partial_x u^\sharp) \\ u \partial_x v^\sharp \\ u \partial_x T^\sharp \end{pmatrix} = \begin{pmatrix} B_1^1(u, u^\sharp) \\ B_1^2(u, v^\sharp) \\ B_1^3(u, T^\sharp) \end{pmatrix} \tag{2.9}$$

and

$$B_2(U,U^\sharp) := \begin{pmatrix} Q(w(u)\partial_z u^\sharp) \\ w(u)\partial_z v^\sharp \\ w(u)\partial_z T^\sharp \end{pmatrix} = \begin{pmatrix} B_2^1(u,u^\sharp) \\ B_2^2(u,v^\sharp) \\ B_2^3(u,T^\sharp) \end{pmatrix}. \tag{2.10}$$

We also set $B^j = B_1^j + B_2^j$, $j = 1,2,3$. We summarize some properties of $B$ needed in the sequel

**Lemma 2.1.** *$B$ is well defined as a bilinear and continuous map from $V \times V$ to $V'$, from $V \times V_{(2)}$ and $V_{(2)} \times V$ to $H$. Moreover $B$ satisfies the following properties and estimates:*

*(i) For any $U, U^\sharp \in V$ and $\langle B(U,U^\sharp), U^\sharp\rangle = 0$.*

*(ii) For $U, U^\sharp, U^\flat \in V$*

$$|\langle B(U,U^\sharp), U^\flat\rangle| \leq c\|U\|\|U^\sharp\||U^\flat|^{1/2}\|U^\flat\|^{1/2}. \tag{2.11}$$

*(iii) On the other hand if we assume that $U \in V$ $U^\sharp \in V_{(2)}$ and $U^\flat \in H$ then*

$$|\langle B(U,U^\sharp), U^\flat\rangle| \leq c\|U\|\|U^\sharp\|^{1/2}|U^\sharp|_{(2)}^{1/2}|U^\flat|. \tag{2.12}$$

*In particular, for $U \in V_{(2)}$,*

$$|B(U,U)|^2 \leq c\|U\|^3|U|_{(2)}. \tag{2.13}$$

*Also if $U = (\mathbf{v},T) = (u,v,T) \in V_{(2)}$, $U^\sharp \in V$ and $U^b \in H$, then*

$$|\langle B(U,U^\sharp), U^\flat\rangle| \leq c\|u\|^{1/2}|u|_{(2)}^{1/2}\|U^\sharp\||U^\flat|. \tag{2.14}$$

*(iv) For $U \in V_{(2)}$, $B(U) \in \tilde{V}$ and satisfies the estimate*

$$\|B(U,U)\|^2 \leq c\|U\||U|_{(2)}^3. \tag{2.15}$$

*(v) Given $U, U^\sharp \in V_{(2)}$, $U^\flat \in H$*

$$|\langle B_1^1(u,u^\sharp), u^\flat\rangle| \leq c|u|^{1/2}|u|_{(2)}^{1/2}|\partial_x u^\sharp||u^\flat|, \tag{2.16}$$

$$|\langle B_1^1(u,u^\sharp), u^\flat\rangle| \leq c|u|^{1/2}\|u\|^{1/2}|\partial_x u^\sharp|^{1/2}\|\partial_x u^\sharp\|^{1/2}|u^\flat|. \tag{2.17}$$

*On the other hand*

$$|\langle B_2^1(u,u^\sharp), u^\flat\rangle| \leq c|\partial_x u||\partial_z u^\sharp|^{1/2}\|\partial_z u^\sharp\|^{1/2}|u^\flat|, \tag{2.18}$$

$$|\langle B_2^1(u,u^\sharp), u^\flat\rangle| \leq c\|u\|^{1/2}|u|_{(2)}^{1/2}|\partial_z u^\sharp||u^\flat|. \tag{2.19}$$

*(vi) For $U = (\mathbf{v}, T) \in D(A)$*

$$\langle B^1(u,u), -\partial_{zz}u\rangle = -\frac{2}{h}\int_{\mathcal{M}} u\partial_x u(\alpha_{\mathbf{v}}u(x,0) + \partial_z u(x,-h)d\mathcal{M} \tag{2.20}$$

*which admits the estimate*

$$|\langle B^1(u,u), -\partial_{zz}u\rangle| \leq c(|u|\|u\|^2 + |\partial_z u|^{1/2}\|\partial_z u\|^{1/2}|u|^{1/2}\|u\|^{3/2}). \tag{2.21}$$

The continuity properties of $B$ as well as the basic cancellation property (i) are well established in the literature. The estimates (2.16), (2.17) may be established as for the classical Navier-Stokes systems (see, for example, [45]). On the other hand the estimates (2.11), (2.12), (2.14), (2.18), (2.19), (2.21) may be proved with anisotropic techniques. See [46] or [23]. The property (2.15), which is new and requires extensive computations, may be found in [21].

We next capture the Coriolis forcing with the bounded operator $E : H \to H$ given by

$$EU := \begin{pmatrix} -Qfv \\ fu \\ 0 \end{pmatrix}. \tag{2.22}$$

We observe that $E$ is also continuous from $V$ to $\tilde{V}$ and that

$$|EU| \leq c|U|, \quad \|EU\| \leq c\|U\|. \tag{2.23}$$

Finally, for brevity of notation we shall sometimes write

$$N(U) = A_p U + B(U,U) + EU, \quad U \in V. \tag{2.24}$$

## 2.4 The stochastic framework: nonlinear, multiplicative white noise forcing

It finally remains to define the white noise driven terms in (1.1). To begin we fix a stochastic basis $\mathcal{S} := (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, \{W^k\}_{k\geq 1})$, that is a filtered probability space with $\{W^k\}_{k\geq 1}$ a sequence of independent standard 1-D Brownian motions relative to the filtration $\mathcal{F}_t$. In order to avoid unnecessary complications below we may assume that $\mathcal{F}_t$ is complete and right continuous (see [15]). Fix a separable Hilbert space $\mathfrak{U}$ with an associated orthonormal basis $\{e_k\}$. We may formally define $W$ by taking $W = \sum_k W^k e_k$. As such $W$ is a cylindrical Brownian motion evolving over $\mathfrak{U}$.

We next recall some basic definitions and properties of spaces of Hilbert-Schmidt operators. For this purpose we suppose that $X$ and $\tilde{X}$ are any separable Hilbert spaces with the associated norms and inner products given by $|\cdot|_X$, $|\cdot|_{\tilde{X}}$ and $\langle\cdot,\cdot\rangle_X$ $\langle\cdot,\cdot\rangle_{\tilde{X}}$, respectively. We denote by

$$L_2(\mathfrak{U}, X) = \{R \in \mathcal{L}(\mathfrak{U}, X) : \sum_k |Re_k|_X^2 < \infty\},$$

the collection of Hilbert Schmidt operators from $\mathfrak{U}$ to $X$. By endowing this collection with the inner product

$$\langle R, S\rangle_{L_2(\mathfrak{U},X)} = \sum_k \langle Re_k, Se_k\rangle_X,$$

we may consider $L_2(\mathfrak{U}, X)$ as itself being a Hilbert space. One may readily show that if $R^{(1)} \in L_2(\mathfrak{U}, X)$ and $R^{(2)} \in L(X, \tilde{X})$ then indeed $R^{(2)}R^{(1)} \in L_2(\mathfrak{U}, \tilde{X})$.

Given an $X$-valued predictable[5] process $G \in L^2(\Omega; L^2_{loc}([0,\infty), L_2(\mathfrak{U}, X)))$ one may define the (Itō) stochastic integral

$$M_t := \int_0^t G dW = \sum_k \int_0^t G_k dW^k,$$

as a square integrable function from $\Omega$ into $X$. Furthermore $M_t$ is an element of $\mathcal{M}^2_X$, that is the space of all $X$-valued square integrable martingales (see [42, Section 2.2, 2.3]), and, as such, $\{M_t\}_{t\geq 0}$ has many desirable properties. Most notably the Burkholder-Davis-Gundy (BDG) inequality holds which in our context takes the form

$$\mathbb{E}\left(\sup_{t'\in[0,t]} \left|\int_0^{t'} G dW\right|_X\right) \leq c\ \mathbb{E}\left(\int_0^t |G|^2_{L_2(\mathfrak{U},X)}\right)^{1/2}, \tag{2.25}$$

for any $t > 0$, where $c$ is here an absolute constant.

Given any Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ we denote by $Bnd_u(\mathcal{X}, \mathcal{Y})$, the collection of all mappings

$$\Psi : \Omega \times [0,\infty) \times \mathcal{X} \to \mathcal{Y},$$

such that $\Psi$ is almost surely continuous in $[0,\infty) \times \mathcal{X}$ and

$$\|\Psi(x)\|_{\mathcal{Y}} \leq c(1 + \|x\|_{\mathcal{X}}), \quad x \in \mathcal{X},$$

where the numerical constant $c$ may be chosen independently of $t$ and $\omega$. If in addition

$$\|\Psi(x) - \Psi(y)\|_{\mathcal{Y}} \leq c\|x - y\|_{\mathcal{X}}, \quad x, y \in \mathcal{X}$$

we say that $\Psi$ is in $Lip_u(\mathcal{X}, \mathcal{Y})$.

With these notations now in place we define

$$\sigma(U) = \begin{pmatrix} Q\sigma_u(\mathbf{v}, T) \\ \sigma_v(\mathbf{v}, T) \\ \sigma_T(\mathbf{v}, T) \end{pmatrix} \tag{2.26}$$

[5] For a given stochastic basis $\mathcal{S}$, let $\Phi = [0,\infty) \times \Omega$ and take $\mathcal{G}$ to be the $\sigma$-algebra generated by sets of the form

$$(s,t] \times F, \quad 0 \leq s < t < \infty, F \in \mathcal{F}_s; \qquad \{0\} \times F, \quad F \in \mathcal{F}_0.$$

Recall that a $X$ valued process $U$ is called predictable (with respect to the stochastic basis $\mathcal{S}$) if it is measurable from $(\Phi, \mathcal{G})$ into $(X, \mathcal{B}(X))$, $\mathcal{B}(X)$ being the family of Borel sets of $X$.

We shall assume throughout this work that

$$\sigma : \Omega \times [0,\infty) \times H \to L_2(\mathfrak{U}, H)$$

such that

$$\begin{aligned}&\text{If } U \text{ is an } H\text{-valued, predictable process, then}\\ &\qquad\qquad \sigma(U) \text{ is an } L_2(\mathfrak{U}, H)\text{-valued, predictable process,}\end{aligned} \tag{2.27}$$

and

$$\sigma \in Lip_u(H, L_2(\mathfrak{U}, H)) \cap Lip_u(V, L_2(\mathfrak{U}, V)) \cap Bnd_u(V, L_2(\mathfrak{U}, D(A))). \tag{2.28}$$

Note that under the conditions imposed above the stochastic integral $\int_0^\tau \sigma(U)dW$ may be shown to be well defined, taking values in $H$ for any $H$ predictable $U \in L^2(\Omega, L^2_{loc}([0,\infty); H))$. Denoting $\sigma_k(\cdot) = \sigma(\cdot)e_k$ we may interpret this integral in the expansion[6]

$$\int_0^t \sigma(U)dW = \sum_{k\geq 1} \int_0^t \sigma^k(U)dW^k = \sum_{k\geq 1} \begin{pmatrix} \int_0^t Q\sigma_u^k(U)dW^k, \\ \int_0^t \sigma_v^k(U)dW^k, \\ \int_0^t \sigma_T^k(U)dW^k \end{pmatrix}.$$

**Remark 2.1.** *The condition (2.28) may be weakened to*

$$\sigma \in Lip_u(H, L_2(\mathfrak{U}, H)) \cap Lip_u(V, L_2(\mathfrak{U}, V)) \cap Bnd_u(D(A), L_2(\mathfrak{U}, D(A))) \tag{2.29}$$

*in the proof of local and maximal existence of solutions below (see Proposition 3.1). However, for the proof of global existence of solutions we need the stronger condition (2.28). See Remark 4.2 below, for further details. Even with this more restrictive condition* (2.28) *the theory covers a physically interesting class of additive and nonlinear multiplicative stochastic forcing regimes relevant to the 'parametrization' problem discussed in the Introduction. We refer the interested reader to [22] for further details and examples.*

For the external forcing terms $F_u, F_v, F_T$ we let:

$$F = \begin{pmatrix} QF_u \\ F_v \\ F_T \end{pmatrix}.$$

[6] To recover the formulation of the stochastic forcings in (1.1), (1.2) we may consider the special case where

$$\begin{aligned}\sigma_u^k &\equiv 0 \text{ when } k = 0\,(mod\,3)\\ \sigma_v^k &\equiv 0 \text{ when } k = 1\,(mod\,3)\\ \sigma_T^k &\equiv 0 \text{ when } k = 2\,(mod\,3)\end{aligned}$$

and take $\dot{W}_1 = \sum_k \dot{W}^{3k} e_{3k}$, $\dot{W}_2 = \sum_k \dot{W}^{3k+1} e_{3k+1}$, $\dot{W}_3 = \sum_k \dot{W}^{3k+1} e_{3k+2}$.

We assume throughout the analysis below that $F$ is an $H$-valued, predictable process with

$$F \in L^2(\Omega; L^2_{loc}([0,\infty), H)). \tag{2.30}$$

We shall allow for the case of probabilistic dependence in the initial data $U_0 = (u_0, v_0, T)$ as well. Specifically we assume that

$$U_0 \in L^2(\Omega; V) \text{ and is } \mathcal{F}_0\text{-measurable.} \tag{2.31}$$

## 2.5 Definition of solutions

With the abstract mathematical definitions for each term in the original system now in hand we may reformulate (2.5) as an abstract evolution equation

$$\begin{aligned} dU + (AU + N(U))dt = Fdt + \sigma(U)dW,\\ U(0) = U_0. \end{aligned} \tag{2.32}$$

More precisely we have the following basic notion of local and global pathwise solutions to the above system.

**Definition 2.1** (Pathwise Strong Solutions of the Primitive Equations)**.** *Let $\mathcal{S} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, W)$ be a fixed stochastic basis. Assume that $F$ is as in (2.30), that $U_0$ satisfies* (2.31) *and that $\sigma$ satisfies (2.27), (2.28).*

*(i) A pair $(U, \tau)$ is* a local strong (pathwise) solution *of (2.32) if $\tau$ is a strictly positive stopping time and $U(\cdot \wedge \tau)$ is a $\mathcal{F}_t$ adapted process in $H$ so that*

$$\begin{aligned} U(\cdot \wedge \tau) \in L^2(\Omega; C([0,\infty); V)),\\ U(\tau)1\!\!1_{t\leq\tau} \in L^2(\Omega; L^2_{loc}([0,\infty); D(A))), \end{aligned} \tag{2.33}$$

*and satisfies, for every $t \geq 0$ and every $\tilde{U} \in H$,*

$$\begin{aligned} \langle U(t\wedge\tau), \tilde{U}\rangle + \int_0^{t\wedge\tau} \langle AU + N(U), \tilde{U}\rangle ds\\ = \langle U_0, \tilde{U}\rangle + \int_0^{t\wedge\tau} \langle F, \tilde{U}\rangle ds + \int_0^{t\wedge\tau} \langle \sigma(U), \tilde{U}\rangle dW. \end{aligned} \tag{2.34}$$

*(ii) Strong solutions of* (2.32) *are said to be* (pathwise) unique *up to a stopping time $\tau > 0$ if given any pair of strong solutions $(U^1, \tau)$, $(U^2, \tau)$ which coincide at $t = 0$ on $\tilde{\Omega} = \{U^1(0) = U^2(0)\}$, then*

$$\mathbb{P}\left(1\!\!1_{\tilde{\Omega}}(U^1(t\wedge\tau) - U^2(t\wedge\tau)) = 0; \forall t \geq 0\right) = 1.$$

*(iii) Suppose that $\{\tau_n\}_{n\geq 1}$ is a strictly increasing sequence of stopping times converging to a (possibly infinite) stopping time $\xi$ and assume that $U$ is a continuous $\mathcal{F}_t$-adapted process in $H$. We say that the triple $(U, \xi, \{\tau_n\}_{n\geq 1})$*

*is* a maximal strong solution *if* $(U, \tau_n)$ *is a local strong solution for each* $n$ *and*

$$\sup_{t\in[0,\xi]} \|U\|^2 + \int_0^\xi |AU|^2 ds = \infty \tag{2.35}$$

*almost surely on the set* $\{\xi < \infty\}$.

*(iv) If* $(U, \xi, \{\tau_n\}_{n\geq 1})$ *is a maximal strong solution and* $\xi = \infty$ *a.s. then we say that the solution is global.*

We now have a complete mathematical framework and may state, in precise terms, the main theorem in this work:

**Theorem 2.1.** *Suppose that the conditions imposed in Definition 2.1 hold. Then there exists a unique global solution* $U$ *of (2.32).*

# 3 Local and Maximal Existence and Uniqueness

The proof of local and maximal existence of solutions for (2.32) makes use of techniques developed for the 3D Navier-Stokes Equations [20]. Since the analysis here is very similar on many points to [20] our treatment will be brief in some details. However, one crucial step, to show that the Galerkin approximations associated to (2.32) are Cauchy (in appropriate spaces) is quite delicate. This is due to stray terms that arise from the discretization which must be controlled. See Proposition 3.2 below.

**Proposition 3.1.** *Suppose that* $U_0$, $F$ *satisfy the conditions imposed in Definition 2.1. For* $\sigma$ *we assume (2.27) and may weaken (2.28) to (2.29). Then there exists a unique maximal strong solution* $(U, \xi)$ *for (2.32). Moreover, for any (deterministic)* $t > 0$,

$$\mathbb{E}\left(\sup_{0\leq t'\leq \xi\wedge t} |U|^2 + \int_0^{\xi\wedge t} \|U\|^2 dt'\right) < \infty. \tag{3.1}$$

*Proof.* The first step in the proof, to establish certain Cauchy estimates for the Galerkin approximations (3.2) of (2.32) is carried out in in Lemma 3.2. For the details of the passage to limit we refer the reader to [20, Proposition 4.2] and the remarks thereafter.

To establish local, pathwise, uniqueness in the sense of Definition 2.1 we note that the estimate (2.11) of $B$ (in dimension 2) is the same as may be achieved for the Navier-Stokes non-linearity in $d = 3$. The proof is therefore identical to [20, Proposition 4.1].

With a local strong solution in hand it remains to extend this solution to a maximal existence time $\xi$ as in Definition 2.1, (iii). For this point we may employ an argument going back to [24]. For a more recent treatment see [20, Lemma 4.1, 4.2, Theorem 4.1]. Since we have the cancellation property in $B$ (Lemma 3.2,(i)) the bound on the weak norms up to a possible finite time blow up (3.1) may be established exactly as in [20, Lemma 4.2] $\square$

## 3.1 Local Cauchy Estimates for the Galerkin System

We turn now to the task of estimating the difference of solutions of the Galerkin system associated to (2.32) at different orders. We begin by recalling some definitions. A $\mathcal{F}_t$-adapted process $U^{(n)} \in L^2(\Omega, C([0,\infty); H_n))$ is a solution of the Galerkin system of order $n$ for (2.32) if it satisfies:

$$\begin{gathered} dU^{(n)} + (AU^{(n)} + P_n N(U^{(n)}))dt = P_n F dt + P_n \sigma(U^{(n)}) dW, \\ U^{(n)}(0) = P_n U_0. \end{gathered} \tag{3.2}$$

Note that by the standard theory of stochastic ordinary differential equations one may establish the global existence of a unique solution $U^{(n)}$ at each order. See e.g. [19] for details.

**Proposition 3.2.** *Let $\{U^{(n)}\}_{n\geq 1}$ be the (global) solutions of the Galerkin systems (3.2) and suppose that there exists a deterministic constant $M$ such that*

$$\|U_0\|^2 \leq M \quad a.s. \tag{3.3}$$

*Then*

*(i) there exists a stopping time $\tau$, with $\tau > 0$, a subsequence $n_j$ and a process $U$ almost surely in $C([0,\infty); V) \cap L^2_{loc}([0,\infty); D(A))$ such that:*

$$\lim_{j\to\infty} \sup_{t\in[0,\tau]} \|U^{(n_j)} - U\|^2 + \int_0^\tau |A(U^{(n_j)} - U)|^2 ds = 0, \tag{3.4}$$

*almost surely.*

*(ii) for any $p \geq 1$, there exists a sequence of $\Omega_{n_j} \in \mathcal{F}_0$, with $\Omega_{n_j} \uparrow \Omega$ such that:*

$$\sup_j \mathbb{E}\left[ \mathbb{1}_{\Omega_{n_j}} \left( \sup_{t\in[0,\tau]} \|U^{(n_j)}\|^2 + \int_0^\tau |AU^{(n_j)}|^2 ds \right)^{p/2} \right] < \infty \tag{3.5}$$

*and*

$$\mathbb{E}\left( \sup_{t\in[0,\tau]} \|U\|^2 + \int_0^\tau |AU|^2 ds \right)^{p/2} < \infty \tag{3.6}$$

**Remark 3.1.** *The technical condition (3.3) is needed so that we may obtain the uniform pathwise bound:*

$$\sup_{m,n} \operatorname*{ess\,sup}_{\omega\in\Omega} \left( \sup_{0\leq t'\leq \tau^M_{m,n}} \|U^{(m)}\|^2 + \int_0^{\tau^M_{m,n}} (1 + |AU^{(m)}|^2) ds \right) < \infty. \tag{3.7}$$

*See (3.8), (3.16) below. Note however that this condition may be removed in the final step of the proof of the local existence. See [20, Proposition 4.2].*

*Proof.* As in previous work [20], the proof consists in establishing the sufficient conditions (3.10), (3.17) for [20, Lemma 5.1] (see also related results in [33]), from which (i) and (ii) follow directly. The proof makes use of some delicate estimates present even in the deterministic case ($\sigma \equiv 0$) that have been carried out in a separate work [21].

We assume with no loss of generality that $M > 1$ and consider the stopping times

$$\tau_n^M = \inf_{t \geq 0} \left\{ \sup_{t' \in [0,t]} \|U^{(n)}\|^2 + \int_0^t |AU^{(n)}|^2 dt' > 4M \right\}. \tag{3.8}$$

Note that (3.8) implies that

$$\sup_{t' \in [0,t]} ||U^{(n)}||^2 + \int_0^t |AU^{(n)}|^2 dt' \leq 4M, \text{ for } 0 \leq t < \tau_n^M. \tag{3.9}$$

We set $\tau_{m,n}^M := \tau_n^M \wedge \tau_m^M$. The first step in the proof is to perform estimates on $U^{(m)} - U^{(n)}$ which we denote by $R^{(m,n)}$ to simplify the notation below. We will show that

$$\lim_{n \to \infty} \sup_{m > n} \mathbb{E} \left( \sup_{0 \leq t' \leq \tau_{m,n}^M} \|R^{(m,n)}\|^2 + \int_0^{\tau_{m,n}^M} |AR^{(m,n)}|^2 dt \right) = 0, \tag{3.10}$$

which is the first condition required for [20, Lemma 5.1].

We fix $m > n$, subtract the equations for $m, n$, then apply $A^{1/2}$ to the resulting system. Note that $D(A^{1/2}) = V$ with $\|U\|^2 = |A^{1/2}U|$. By the Itō lemma we may also infer that

$$\begin{aligned} d\|R^{(m,n)}\|^2 + & 2|AR^{(m,n)}|^2 dt \\ = & -2\langle P_m N(U^{(m)}) - P_n N(U^{(n)}), AR^{(m,n)} \rangle dt \\ & + 2\langle P_m^n F, AR^{(m,n)} \rangle dt \\ & + \|P_m \sigma(U^{(m)}) - P_n \sigma(U^{(n)})\|_{L_2(\mathfrak{U}, V)}^2 dt \\ & + 2\langle P_m \sigma(U^{(m)}) - P_n \sigma(U^{(n)}), AR^{(m,n)} \rangle dW. \end{aligned} \tag{3.11}$$

We now estimate each of the terms above with a view of finally applying a stochastic analogue of the Gronwall inequality, [20, Lemma 5.3]. With this in mind fix any pair of stopping times $\tau_a, \tau_b$ such that $0 \leq \tau_a \leq \tau_b \leq \tau_{n,m}^M$. By integrating the above system, taking a supremum over the random interval

$[\tau_a,\tau_b]$ and finally taking an expected values we may infer that

$$
\begin{aligned}
\mathbb{E}&\left(\sup_{t\in[\tau_a,\tau_b]}\|R^{(m,n)}\|^2+\int_{\tau_a}^{\tau_b}|AR^{(m,n)}|^2dt\right)\\
&\leq c\mathbb{E}\|R^{(m,n)}(\tau_a)\|^2+c\mathbb{E}\int_{\tau_a}^{\tau_b}|\langle (P_m-P_n)F,AR^{(m,n)}\rangle|dt\\
&\quad+c\mathbb{E}\int_{\tau_a}^{\tau_b}|\langle P_mN(U^{(m)})-P_nN(U^{(n)}),AR^{(m,n)}\rangle|dt\\
&\quad+c\mathbb{E}\int_{\tau_a}^{\tau_b}\|P_m\sigma(U^{(m)})-P_n\sigma(U^{(n)})\|^2_{L_2(\mathfrak{U},V)}dt\\
&\quad+c\mathbb{E}\sup_{t\in[\tau_a,\tau_b]}\left|\int_{\tau_a}^{t}\langle P_m\sigma(U^{(m)})-P_n\sigma(U^{(n)}),AR^{(m,n)}\rangle dW\right|.
\end{aligned}
\tag{3.12}
$$

We begin by addressing the 'deterministic portions' of (3.12). Using the equivalence fractional order spaces, (2.15) and the generalized Poincaré inequality it is shown in [21], (see (3.13) in Theorem 3.1 of [21]) that:

$$
\begin{aligned}
|\langle P_mN(U^{(m)})&-P_nN(U^{(n)}),AR^{(m,n)}\rangle|\\
&\leq\frac{1}{2}|AR^{(m,n)}|^2+c(1+|AU^{(m)}|^2+\|U^{(n)}\|^4)\|R^{(m,n)}\|^2\\
&\quad+\frac{c}{\lambda_n^{1/4}}(1+\|U^{(n)}\|^2)(1+|AU^{(n)}|^2).
\end{aligned}
\tag{3.13}
$$

We next consider the terms which arise only in the stochastic context. The Itō correction term may be estimated according to

$$
\begin{aligned}
\|P_m\sigma(U^{(m)})&-P_n\sigma(U^{(n)})\|^2_{L_2(\mathfrak{U},V)}\\
&\leq c\left(\|\sigma(U^{(m)})-\sigma(U^{(n)})\|^2_{L_2(\mathfrak{U},V)}+\|Q_n\sigma(U^{(n)})\|^2_{L_2(\mathfrak{U},V)}\right)\\
&\leq c(\|R^{m,n}\|^2+\frac{1}{\lambda_n}|A\sigma(U^{(n)})|^2_{L_2(\mathfrak{U},H)})\\
&\leq c\left(\|R^{m,n}\|^2+\frac{1}{\lambda_n}(1+|AU^{(n)}|^2)\right)
\end{aligned}
\tag{3.14}
$$

For the second inequality we have made use of the generalized Poincaré Inequality[7]. The final inequality follows from (2.29). For the stochastic integral terms

[7]We use the special case $\|Q_nU^\sharp\|^2\leq\frac{1}{\lambda_n}|AU^\sharp|^2$, which holds for any $U^\sharp\in D(A)$.

we apply (2.25) and deduce

$$
\begin{aligned}
&\mathbb{E} \sup_{\tau_a \le t' \le \tau_b} \left| \int_{\tau_a}^{t'} \langle P_m \sigma(U^{(m)}) - P_n \sigma(U^{(n)}), AR^{(m,n)} \rangle dW \right| \\
\le & c\mathbb{E} \left( \int_{\tau_a}^{\tau_b} \langle P_m \sigma(U^{(m)}) - P_n \sigma(U^{(n)}), AR^{(m,n)} \rangle^2_{L_2(\mathfrak{U},H)} dt' \right)^{1/2} \\
\le & c\mathbb{E} \left( \int_{\tau_a}^{\tau_b} \| P_m \sigma(U^{(m)}) - P_n \sigma(U^{(n)}) \|^2_{L_2(\mathfrak{U},V)} \| R^{(m,n)} \|^2 dt' \right)^{1/2} \\
\le & c\mathbb{E} \left( \sup_{t \in [\tau_a, \tau_b]} \| R^{(m,n)} \| \right. \\
& \qquad \left. \cdot \left( \int_{\tau_a}^{\tau_b} \| P_m \sigma(U^{(m)}) - P_n \sigma(U^{(n)}) \|^2_{L_2(\mathfrak{U},V)} dt' \right)^{1/2} \right) \\
\le & \frac{1}{2} \mathbb{E} \left( \sup_{t \in [\tau_a, \tau_b]} \| R^{(m,n)} \|^2 \right) \\
& + c\mathbb{E} \left( \int_{\tau_a}^{\tau_b} (\| R^{(m,n)} \|^2 + \frac{1}{\lambda_n} (1 + |AU^{(n)}|^2)) dt' \right).
\end{aligned} \tag{3.15}
$$

The last inequality is achieved by applying the Schwarz inequality and then (3.14).

We now gather the estimates (3.13), (3.14)), (3.15) and compare with (3.12). Since $0 \le \tau_a \le \tau_b \le \tau^M_{m,n}$ we conclude, using (3.9) that

$$
\begin{aligned}
&\mathbb{E} \left( \sup_{t \in [\tau_a, \tau_b]} \| R^{(m,n)} \|^2 + \int_{\tau_a}^{\tau_b} |AR^{(m,n)}|^2 dt \right) \\
\le & c \ \mathbb{E} \| R^{(m,n} (\tau_a) \|^2 \\
& + c \ \mathbb{E} \int_{\tau_a}^{\tau_b} \left( (1 + |AU^{(m)}|^2) \| R^{(m,n)} \|^2 + \frac{1}{\lambda_n^{1/4}} (1 + |AU^{(n)}|^2) + |Q_n F|^2 \right) dt.
\end{aligned} \tag{3.16}
$$

Observe that the generic constant $c$ is independent of $m, n$ and that (3.7) (3.10) now follows from the stochastic Gronwall lemma.

It remains to establish the other requirement of [20, Lemma 5.1]. In the present context this translates to

$$
\lim_{\delta \to 0} \sup_n \mathbb{P} \left( \sup_{0 \le t' \le \tau_n^M \wedge \delta} \| U^{(n)} \|^2 + \int_0^{\tau_n^M \wedge \delta} |AU^{(n)}|^2 dt' > \tilde{M} \right) = 0, \tag{3.17}
$$

for every $\tilde{M} > M$. By applying Itō we infer an equation for $t \mapsto \| U^{(n)}(t) \|^2$ very similar to (4.41), below. Since, as for the Navier-Stokes system in $d = 3$ (see (2.12))

$$
|\langle B(U), AU \rangle| \le \| U \|^{3/2} |AU|^{3/2}, \quad U \in D(A)
$$

and since the $A_p$ and $E$ terms are lower order (see (2.7),(2.23)), we may establish (3.17) with a direct application of Doob's inequality exactly as in [20, Proposition 3.1]. With (3.17), the proof is complete. $\square$

# 4 Global Existence

We now implement a series of anisotropic estimates that are used to infer global existence. Due to the non-commutativity introduced by the physical boundary conditions we must first define a new variable $\hat{U}$ that satisfies a system obeying the rules of ordinary calculus. We are then are able to derive suitable estimates for $\hat{u}_z$ and then $\hat{u}_x$ and finally for the entire original system in $V$. Since the resulting estimates yield only pathwise (rather than moment) bounds we must finally recourse to some involved stopping time arguments which make essential use of Lemma 5.1.

## 4.1 A Change of Variable to a Random PDE and some Auxiliary Estimates

We consider the linear stochastic partial differential equation

$$\partial_t \check{u} - \nu\Delta\check{u} + \partial_x \check{p}_s = \mathbb{1}_{t\leq\xi}\sigma_u(\mathbf{v},T)\dot{W}_1, \tag{4.1a}$$
$$\partial_t \check{v} - \nu\Delta\check{v} = \mathbb{1}_{t\leq\xi}\sigma_v(\mathbf{v},T)\dot{W}_2, \tag{4.1b}$$
$$\partial_t \check{T} - \mu\Delta\check{T} = \mathbb{1}_{t\leq\xi}\sigma_T(\mathbf{v},T)\dot{W}_3, \tag{4.1c}$$

with $\xi$ as in Proposition 3.1. This system is supplemented with the same boundary conditions as in (1.3). We posit the zero initial condition $\check{u}(0) = \check{v}(0) = \check{T}(0) = 0$. Note that the stochastic forcing terms depend on $(U,\xi) = ((\mathbf{v},T),\xi)$, maximal strong solution solution we found for (1.1)-(1.4) in Proposition 3.1; $\sigma$ is exactly the same as appearing in (1.1) and in particular satisfies (2.28). As in Section 2.5, (4.1) may be formulated in an abstract form:

$$d\check{U} + A\check{U}dt = \mathbb{1}_{t\leq\xi}\sigma(U)dW, \quad \check{U}(0) = 0. \tag{4.2}$$

We shall need the following preliminary estimates below for $\check{U}$.

**Lemma 4.1.** *There exists a unique global pathwise strong solution of (4.2) taking its values in $D(A)$. Additionally for any deterministic finite time $t > 0$, we have*

$$\mathbb{E}\left(\sup_{t'\in[0,t]} |A\check{U}|^2\right) < \infty. \tag{4.3}$$

*Proof.* We briefly outline the formal estimates that lead to (4.3). Since (4.2) is linear in the unknown everything, including the global existence, may be easily justified with a suitable Galerkin scheme (see e.g. [19]).

Formally then we multiply (4.2) by $A$ and apply the Itō lemma in $H$ to deduce

$$\begin{aligned} &d|A\check{U}|^2 + 2|A^{3/2}\check{U}|^2 dt \\ &\qquad\qquad = 2\mathbb{1}_{t\leq\xi}\langle A\sigma(U), A\check{U}\rangle dW + \mathbb{1}_{t\leq\xi}|A\sigma(U)|^2_{L_2(\mathfrak{U},H)}dt. \end{aligned} \tag{4.4}$$

Fixing arbitrary $t > 0$ and taking a supremum over $t' \leq t$ and then expected values we infer from (2.29), (4.4) and the fact that $\tilde{U}(0) = 0$,

$$\begin{aligned} &\mathbb{E}\left(\sup_{t'\in[0,t]}|A\check{U}|^2\right) \\ &\leq \mathbb{E}\sup_{t'\in[0,t]}\left|\int_0^{t'\wedge\xi}\langle A\sigma(U), A\check{U}\rangle dW\right| + \mathbb{E}\int_0^{t\wedge\xi}|\sigma(U)|^2_{L_2(\mathfrak{U},D(A))}dt' \\ &\leq \frac{1}{2}\mathbb{E}\left(\sup_{t'\in[0,t]}|A\check{U}|^2\right) + c\mathbb{E}\int_0^{t\wedge\xi}|\sigma(U)|^2_{L_2(\mathfrak{U},D(A)}dt' \\ &\leq \frac{1}{2}\mathbb{E}\left(\sup_{t'\in[0,t]}|A\check{U}|^2\right) + c\mathbb{E}\int_0^{t\wedge\xi}(1+\|U\|^2)dt'. \end{aligned} \tag{4.5}$$

For the stochastic integral terms after the first inequality we apply (2.25) and then estimate in a similar manner to (3.15). The final inequality is a consequence of the assumption (2.28) imposed on $\sigma$. To complete the proof we rearrange (4.5) and refer to (3.1) in Proposition 3.1 to conclude (4.3). □

We next subtract (4.1a) from (2.5) and define $\hat{U} = U - \check{U}$. On the random interval $[0,\xi)$ we see that $\hat{U}$ must satisfy the following partial differential equation (without white noise driven forcing but with random coefficients)

$$\frac{d}{dt}\hat{U} + A\hat{U} + A_p(\hat{U}+\check{U}) + B(\hat{U}+\check{U}) + E(\hat{U}+\check{U}) = F. \tag{4.6}$$

Note that, in contrast to (2.32) this new system satisfies the usual rules of ordinary calculus.

We may rewrite (4.6) in a form more convenient for our purposes below:

$$\begin{aligned} &\frac{d}{dt}\hat{U} + A\hat{U} + A_p\hat{U} + B(\hat{U}) + E\hat{U} \\ &\qquad = F - B(\check{U},\check{U}) - B(\check{U},\hat{U}) - B(\hat{U},\check{U}) - E\check{U} - A_p\check{U}. \end{aligned} \tag{4.7}$$

By combining Lemma 4.1 with Proposition 3.1 we may directly infer that

**Lemma 4.2.** *For any deterministic, finite $t > 0$ we have:*

$$\mathbb{E}\left(\sup_{0\leq t'\leq\xi\wedge t}|\hat{U}|^2 + \int_0^{\xi\wedge t}\|\hat{U}\|^2 ds\right) < \infty \tag{4.8}$$

Finally we note that the first momentum equation included in (4.7), which will be the focus of our attention in the subsequent sections, is given by

$$
\begin{aligned}
&\partial_t \hat{u} + \hat{u}\partial_x \hat{u} + w(\hat{u})\partial_z \hat{u} - \nu\Delta\hat{u} - f\hat{v} + \partial_x \hat{p}_s - \beta_T g\rho_0 \int_z^0 \partial_x \hat{T} d\bar{z} \\
&=F_u + f\check{v} + \beta_T g\rho_0 \int_z^0 \partial_x \check{T} d\bar{z} \\
&\quad - (\check{u}\partial_x \check{u} + w(\check{u})\partial_z \check{u}) - (\check{u}\partial_x \hat{u} + w(\check{u})\partial_z \hat{u}) - (\hat{u}\partial_x \check{u} + w(\hat{u})\partial_z \check{u}) \\
&=F_u + f\check{v} + \beta_T g\rho_0 \int_z^0 \partial_x \check{T} d\bar{z} - (\tilde{\mathcal{B}}^1(\check{u},\check{u}) + \tilde{\mathcal{B}}^1(\check{u},\hat{u}) + \tilde{\mathcal{B}}^1(\hat{u},\check{u})).
\end{aligned}
\tag{4.9}
$$

**Remark 4.1.** *We infer from (4.8) that,*

$$
\sup_{0 \le t' \le \xi\wedge t} |\hat{U}|^2 + \int_0^{\xi\wedge t} ||\hat{U}||^2 ds \le K^1(t,\omega) < \infty \tag{4.10}
$$

*where here and below, $K, K^i$, denote a.s. finite constants which depend on $t$, on the data such as norms of $U_0$, $F$ and on $\omega$ through these norms and though stochastic integral terms driven by $W$.*

## 4.2 Anisotropic Estimates

We now turn to the estimates for $\partial_z \hat{u}$.

**Lemma 4.3.** *Let $(U,\xi) = ((u,v,T),\xi)$ be the unique maximal strong solution of (2.32) guaranteed by Proposition 3.1. Then, for every $t > 0$ there exists a finite constant $K = K^2(t,\omega) < \infty$ depending on $t,\omega$ and the data such that*

$$
\sup_{0 \le t' \le \xi\wedge t} |\partial_z \hat{u}|^2 + \int_0^{\xi\wedge t} \|\partial_z \hat{u}\|^2 ds \le K^2 \quad a.s. \tag{4.11}
$$

*Proof.* We multiply (4.9) by $-Q\partial_{zz}\hat{u}$ and integrate over the domain $\mathcal{M}$. Following closely the computations in [39] we to deduce:

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}&\left(|\partial_z \hat{u}|^2 + \alpha_{\mathbf{v}}|\hat{u}|^2_{L^2(\Gamma_i)}\right) + \nu\|\partial_z\hat{u}\|^2 + \nu\alpha_{\mathbf{v}}|\partial_x\hat{u}|^2_{L^2(\Gamma_i)} \\
&=|P\partial_{zz}\hat{u}|^2 - \int_{\mathcal{M}} F_u Q\partial_{zz}\hat{u}\, d\mathcal{M} \\
&\quad - \beta_T g\rho_0 \int_{\mathcal{M}} \left(\int_z^0 \partial_x(\hat{T} + \check{T}) d\bar{z}\right) Q\partial_{zz}\hat{u}\, d\mathcal{M} \\
&\quad - \int_{\mathcal{M}} f(\hat{v} + \check{v}) Q\partial_{zz}\hat{u}\, d\mathcal{M} \\
&\quad + \frac{2}{h}\int_{\mathcal{M}} \hat{u}\partial_x\hat{u}\left[\alpha_{\mathbf{v}}\hat{u}(0,x) + \partial_z\hat{u}(x,-h)\right] d\mathcal{M} \\
&\quad + \int_{\mathcal{M}} (B^1(\check{u},\check{u}) + B^1(\check{u},\hat{u}) + B^1(\hat{u},\check{u}))\partial_{zz}\hat{u} d\mathcal{M} \\
&\quad = J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7 + J_8.
\end{aligned}
\tag{4.12}
$$

Here the bottom boundary is flat which causes several terms to disappear present in [39]. The term

$$-\langle B^1(\hat{u},\hat{u}), -\partial_{zz}\hat{u}\rangle = \int (\hat{u}\partial_x\hat{u} + w(\hat{u})\partial_z\hat{u}) Q\partial_{zz}\hat{u}\, d\mathcal{M}$$

largely cancels and appears as $J_5$ due to Lemma 2.1, (vi) above. Also we observe that $Q\partial_x\hat{p}_s = 0$ which is why we multiply (4.9) by $-Q\partial_{zz}\hat{u}$ rather than $-\partial_{zz}u$.

The first term $J_1$ on the right hand side of (4.12) reduces to two terms at $z = -h$ and 0 that are estimated using the trace theorem:

$$|J_1| \leq c\|\hat{U}\|^2 + \frac{\nu}{16}\|\partial_z\hat{u}\|^2. \tag{4.13}$$

The estimates for the next three terms are direct:

$$\begin{aligned}
|J_2| \leq& c|F|^2 + \frac{\nu}{16}\|\partial_z\hat{u}\|^2,\\
|J_3| \leq& c(\|\hat{U}\|^2 + \|\check{U}\|^2) + \frac{\nu}{16}\|\partial_z\hat{u}\|^2\\
\leq& c(\|\hat{U}\|^2 + |\check{U}|^2_{(2)}) + \frac{\nu}{16}\|\partial_z\hat{u}\|^2,\\
|J_4| \leq& c(\|\hat{U}\|^2 + \|\check{U}\|^2) + \frac{\nu}{16}\|\partial_z\hat{u}\|^2\\
\leq& c(\|\hat{U}\|^2 + |\check{U}|^2_{(2)}) + \frac{\nu}{16}\|\partial_z\hat{u}\|^2.
\end{aligned} \tag{4.14}$$

For $J_5$ we may estimate using (2.21) and Young's inequality

$$\begin{aligned}
|J_5| \leq& c(|\hat{u}|\|\hat{u}\|^2 + |\partial_z\hat{u}|^{1/2}\|\partial_z\hat{u}\|^{1/2}|\hat{u}|^{1/2}\|\hat{u}\|^{3/2})\\
\leq& c(|\hat{U}|\|\hat{U}\|^2 + |\partial_z\hat{u}|^{2/3}|\hat{U}|^{2/3}\|\hat{U}\|^2) + \frac{\nu}{16}\|\partial_z\hat{u}\|^2\\
\leq& c(|\hat{U}|\|\hat{U}\|^2 + |\partial_z\hat{u}|^2\|\hat{U}\|^2) + \frac{\nu}{16}\|\partial_z\hat{u}\|^2.
\end{aligned} \tag{4.15}$$

For $J_6$, (2.16), (2.19) allow

$$\begin{aligned}
|J_6| \leq& c(|\check{u}|^{1/2}\|\check{u}\| + \|\check{u}\|^{3/2})|\check{u}|^{1/2}_{(2)}|\partial_{zz}\hat{u}|\\
\leq& c\|\check{u}\||\check{u}|_{(2)}|\partial_{zz}\hat{u}|\\
\leq& c\|\check{U}\|^2|\check{U}|^2_{(2)} + \frac{\nu}{16}\|\partial_z\hat{u}\|^2.
\end{aligned} \tag{4.16}$$

For $J_7$ we estimate with (2.16) and (2.19):

$$\begin{aligned}
|J_7| \leq& c(|\check{u}|^{1/2}|\check{u}|^{1/2}_{(2)}\|\hat{u}\| + \|\check{u}\|^{1/2}|\check{u}|^{1/2}_{(2)}|\partial_z\hat{u}|)|\partial_{zz}\hat{u}|\\
\leq& c|\check{U}|^2_{(2)}(\|\hat{U}\|^2 + |\partial_z\hat{u}|^2) + \frac{\nu}{16}\|\partial_z\hat{u}\|^2.
\end{aligned} \tag{4.17}$$

Finally concerning $J_8 = \langle B^1_1(\hat{u},\check{u}) + B^1_2(\hat{u},\check{u}), \partial_{zz}\hat{u}\rangle := J_{8,1} + J_{8,2}$ we estimate

$$\begin{aligned}
|J_{8,1}| \leq& c|\hat{u}|^{1/2}\|\hat{u}\|^{1/2}\|\check{u}\|^{1/2}|\check{u}|^{1/2}_{(2)}\|\partial_z\hat{u}\|\\
\leq& c(|\hat{U}|^2\|\hat{U}\|^2 + \|\check{U}\|^2|\check{U}|^2_{(2)}) + \frac{\nu}{32}\|\partial_z\hat{u}\|^2,
\end{aligned} \tag{4.18}$$

using (2.17), and,

$$
\begin{aligned}
|J_{8,2}| &\leq c\|\hat{u}\|\|\check{u}\|^{1/2}|\check{u}|_{(2)}^{1/2}\|\partial_z\hat{u}\| \\
&\leq c\|\hat{U}\|^2|\check{U}|_{(2)}^2 + \frac{\nu}{32}\|\partial_z\hat{u}\|^2.
\end{aligned}
\tag{4.19}
$$

thanks to (2.18).

Collecting the estimates (4.13), (4.14), (4.15), (4.16), (4.17), (4.18), (4.19) above we may finally observe that

$$
\begin{aligned}
&\frac{d}{dt}\big(|\partial_z\hat{u}|^2 + \alpha_{\mathbf{v}}|\hat{u}|^2_{L^2(\Gamma_i)}\big) + \nu\|\partial_z\hat{u}\|^2 \\
&\quad\leq c(\|\hat{U}\|^2 + |\check{U}|^2_{(2)})|\partial_z\hat{u}|^2 \\
&\qquad + c(1+|\hat{U}|^2)\|\hat{U}\|^2 + c(1+\|\hat{U}\|^2 + \|\check{U}\|^2)|\check{U}|^2_{(2)} + c|F|^2.
\end{aligned}
\tag{4.20}
$$

We therefore conclude that

$$
\frac{d}{dt}\big(|\partial_z\hat{u}|^2 + \alpha_{\mathbf{v}}|\hat{u}|^2_{L^2(\Gamma_i)}\big) \leq (|\partial_z\hat{u}|^2 + \alpha_{\mathbf{v}}|\hat{u}|^2_{L^2(\Gamma_i)})R_1 + R_2 + C|F|^2,
\tag{4.21}
$$

where

$$
\begin{aligned}
R_1 &:= \|\hat{U}\|^2 + |\check{U}|^2_{(2)} \\
R_2 &:= c(1+|\hat{U}|^2)\|\hat{U}\|^2 + c(1+\|\hat{U}\|^2 + \|\check{U}\|^2)|\check{U}|^2_{(2)}
\end{aligned}
\tag{4.22}
$$

and the constants $c$ are as in (4.20). Note that, due to (4.10) and (4.3), for all $t > 0$, there exists a constant $K = K(t,\omega)$ such that,

$$
\int_0^{t\wedge\xi} R_j ds \leq K(t,\omega) < \infty \;\; a.s. \quad j = 1,2.
\tag{4.23}
$$

The (deterministic) Gronwall inequality now yields

$$
\begin{aligned}
\sup_{t'\in[0,\tau_n\wedge t]}|\partial_z\hat{u}|^2 &\leq \sup_{t'\in[0,\tau_n\wedge t]}(|\partial_z\hat{u}|^2 + \alpha_{\mathbf{v}}|\hat{u}|^2_{L^2(\Gamma_i)}) \\
&\leq \exp\left(\int_0^{\xi\wedge t} R_1 dt'\right)\left(|\partial_z u_0|^2 + \int_0^{\xi\wedge t}(R_2 + C|F|^2)dt'\right) \\
&\leq K(t,\omega)\left(1 + \|U_0\|^2 + \int_0^{\xi\wedge t}|F|^2 dt'\right).
\end{aligned}
\tag{4.24}
$$

Finally, returning to (4.20), integrating over $[0,\tau_n\wedge t]$, and then neglecting the terms $|\partial_z\hat{u}|^2 + \alpha_{\mathbf{v}}|\hat{u}|^2_{L^2(\Gamma_i)}$ appearing on the left hand side of the resulting expression, we observe that:

$$
\begin{aligned}
\int_0^{\xi\wedge t}\|\partial_z\hat{u}\|^2 dt' &\leq \|U_0\|^2 + \int_0^{\xi\wedge t}(|\partial_z\hat{u}|^2 R_1 + R_2 + c|F|^2)dt' \\
&\leq K(t,\omega).
\end{aligned}
\tag{4.25}
$$

Combining (4.24) and (4.25), completes the proof. □

We next come to the estimates for $\partial_x u$. Here we show

**Lemma 4.4.** *The hypotheses are the same as in Lemma 4.3. Then, for every $t > 0$, there exists a finite constant $K = K^3(t, \omega) < \infty$ depending on $t, \omega$ and the data such that*

$$\sup_{0 \le t' \le \xi \wedge t} |\partial_x \hat{u}|^2 + \int_0^{\xi \wedge t} \|\partial_x \hat{u}\|^2 dt' \le K^3 \quad a.s. \tag{4.26}$$

*Proof.* The hypotheses being the same as for Lemma 4.3, the conclusions of that Lemma thus hold, and in particular (4.11).

To determine an evolution equation for $|\partial_x \hat{u}|$ we multiply (4.9) by $-\partial_{xx} u$ and integrate over $\mathcal{M}$. After some direct manipulations, this yields

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} |\partial_x \hat{u}|^2 + \nu \|\partial_x \hat{u}\|^2 + \nu \alpha_{\mathbf{v}} |\partial_x \hat{u}|^2_{L^2(\Gamma_i)} \\
&= \beta_T g \rho_0 \int_{\mathcal{M}} \left( \int_z^0 \partial_x (\hat{T} + \check{T}) d\bar{z} \right) \partial_{xx} \hat{u} \, d\mathcal{M} \\
&\quad - \int_{\mathcal{M}} F_u \partial_{xx} \hat{u} \, d\mathcal{M} \\
&\quad - \int_{\mathcal{M}} 2 f (\hat{v} + \check{v}) \partial_{xx} \hat{u} \, d\mathcal{M} \\
&\quad + \int_{\mathcal{M}} (B^1(\hat{u}, \hat{u}) + B^1(\check{u}, \check{u}) + B^1(\check{u}, \hat{u}) + B^1(\hat{u}, \check{u})) \partial_{xx} \hat{u} \, d\mathcal{M} \\
&= J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7.
\end{aligned} \tag{4.27}$$

Notice that in this case the pressure term disappears by integration in $z$, since $P \partial_{xx} \hat{u} = 0$

As above the first three terms are direct

$$\begin{aligned}
|J_1| &\le c(\|\hat{U}\|^2 + \|\check{U}\|^2) + \frac{\nu}{14} \|\partial_x \hat{u}\|^2, \\
|J_2| &\le c|F|^2 + \frac{\nu}{14} \|\partial_x \hat{u}\|^2, \\
|J_3| &\le c(\|\hat{U}\|^2 + \|\check{U}\|^2) + \frac{\nu}{14} \|\partial_x \hat{u}\|^2.
\end{aligned} \tag{4.28}$$

We may handle the term $J_4$ as in [46], however we may also directly apply Lemma 2.1, (2.17), (2.18) to infer

$$\begin{aligned}
|J_4| &\le c(|\hat{u}|^{1/2} \|\hat{u}\|^{1/2} |\partial_x \hat{u}|^{1/2} \|\partial_x \hat{u}\|^{3/2} + |\partial_x \hat{u}| |\partial_z \hat{u}|^{1/2} \|\partial_z \hat{u}\|^{1/2} \|\partial_x \hat{u}\|) \\
&\le c(|\hat{u}|^2 \|\hat{u}\|^2 |\partial_x \hat{u}|^2 + |\partial_x \hat{u}|^2 |\partial_z \hat{u}| \|\partial_z \hat{u}\|) + \frac{\nu}{14} \|\partial_x \hat{u}\|^2 \\
&\le c(|\hat{U}|^2 \|\hat{U}\|^2 + \|\partial_z \hat{u}\|^2) |\partial_x \hat{u}|^2 + \frac{\nu}{14} \|\partial_x \hat{u}\|^2.
\end{aligned} \tag{4.29}$$

The estimates (2.16) - (2.19) allow us to treat the remaining terms $J_5, J_6, J_7$ as well. Indeed

$$\begin{aligned}
|J_5| &\le c(|\check{u}|^{1/2} |\check{u}|^{1/2}_{(2)} \|\check{u}\| + \|\check{u}\|^{3/2} |\check{u}|^{1/2}_{(2)}) \|\partial_x \hat{u}\| \\
&\le c \|\check{U}\|^2 |\check{U}|^2_{(2)} + \frac{\nu}{14} \|\partial_x \hat{u}\|^2.
\end{aligned} \tag{4.30}$$

Also

$$
\begin{aligned}
|J_6| \leq & c(|\check{u}|^{1/2}|\check{u}|_{(2)}^{1/2}|\partial_x \hat{u}| + \|\check{u}\|^{1/2}|\check{u}|_{(2)}^{1/2}|\partial_z \hat{u}|)\|\partial_x \hat{u}\| \\
\leq & c|\check{U}|_{(2)}^2(|\partial_x \hat{u}|^2 + |\partial_z \hat{u}|^2) + \frac{\nu}{14}\|\partial_x \hat{u}\|^2 \\
\leq & c\|\hat{U}\|^2|\check{U}|_{(2)}^2 + \frac{\nu}{14}\|\partial_x \hat{u}\|^2.
\end{aligned} \tag{4.31}
$$

Finally

$$
\begin{aligned}
|J_7| &\leq c(|\hat{u}|^{1/2}\|\hat{u}\|^{1/2}\|\check{u}\|^{1/2}|\check{u}|_{(2)}^{1/2} + |\partial_x \hat{u}|\|\check{u}\|^{1/2}|\check{u}|_{(2)}^{1/2})\|\partial_x \hat{u}\| \\
&\leq c\|\check{u}\||\check{u}|_{(2)}(|\hat{u}|\|\hat{u}\| + |\partial_x \hat{u}|^2) + \frac{\nu}{14}\|\partial_x \hat{u}\|^2 \\
&\leq c\|\hat{U}\|^2|\check{U}|_{(2)}^2 + \frac{\nu}{14}\|\partial_x \hat{u}\|^2.
\end{aligned} \tag{4.32}
$$

Gathering the estimates above, we conclude that:

$$
\begin{aligned}
&\frac{d}{dt}|\partial_x \hat{u}|^2 + \nu\|\partial_x \hat{u}\|^2 \\
&\qquad \leq c(|\hat{U}|^2\|\hat{U}\|^2 + \|\partial_z \hat{u}\|^2)|\partial_x \hat{u}|^2 \\
&\qquad\quad + c(\|\hat{U}\|^2 + \|\check{U}\|^2 + \|\check{U}\|^2|\check{U}|_{(2)}^2 + \|\hat{U}\|^2|\check{U}|_{(2)}^2) + c|F|^2 \\
&\qquad \leq R_3|\partial_x \hat{u}|^2 + R_4 + c|F|^2,
\end{aligned} \tag{4.33}
$$

where $R_3 := c(|\hat{U}|^2\|\hat{U}\|^2 + \|\partial_z \hat{u}\|^2)$ and $R_4 := c(\|\hat{U}\|^2 + \|\check{U}\|^2 + \|\check{U}\|^2|\check{U}|_{(2)}^2 + \|\hat{U}\|^2|\check{U}|_{(2)}^2)$. Dropping the term $\nu\|\partial_x \hat{u}\|^2$, applying the Gronwall inequality and then making use of the assumed bound (4.26) we infer, using (4.10), (4.3) and (4.11), that

$$
\begin{aligned}
\sup_{0 \leq t' \leq \xi \wedge t} |\partial_x \hat{u}|^2 \leq & \exp\left(\int_0^{\xi \wedge t} R_3 dt'\right)\left(|\partial_x u_0|^2 + \int_0^{\xi \wedge t}(R_4 + C|F|^2)dt'\right) \\
\leq & K(t, \omega) < +\infty.
\end{aligned} \tag{4.34}
$$

We then integrate (4.33) from $0, \xi \wedge t$ and infer, using again (4.10), (4.3) and (4.11), that

$$
\begin{aligned}
\int_0^{\xi \wedge t} \|\partial_x \hat{u}\|^2 dt' &\leq \|U_0\|^2 + \int_0^{\xi \wedge t}(R_3|\partial_x \hat{u}|^2 + R_4 + |F|^2)dt' \\
&\leq K(t, \omega) < \infty,
\end{aligned} \tag{4.35}
$$

where the final inequality follows from the previous bound (4.34). This completes the proof of Lemma 4.4. $\square$

**Remark 4.2.** *With some minor modifications to the proof, Lemma 4.4 may established if we merely assume that,*

$$
\sup_{t' \leq \tau_n}\left(|\hat{U}|^2 + \|\check{U}\|^2 + |\partial_z \hat{u}|^2\right) + \int_0^{\tau_n}(\|\hat{U}\|^2 + |\check{U}|_{(2)}^2 + \|\partial_z \hat{u}\|^2)dt' \leq K < \infty \quad a.s. \tag{4.36}
$$

*On the other hand the proof of Lemma 4.3 seems to require that*

$$\sup_{t'\leq\xi} |\check{U}|_{(2)}^2 \leq K < \infty \tag{4.37}$$

*This condition is needed to order handle $J_8$ appearing in (4.12). The requirement (4.37) is achieved due to (4.3) but at the cost of a slightly more restrictive condition on $\sigma$, (2.28), as compared to previous work. We underline here that this is the only point in this work where we require the final condition in (2.28).*

**Remark 4.3.** *We observe that the $H^1$ -norm $||\varphi||$ of a function $\varphi$ is equivalent to the norm $(|\varphi|^2 + |\partial_x\varphi|^2 + |\partial_z\varphi|^2)^{1/2}$, and the $H^2$ - norm $|\varphi|_{(2)}$ of $\varphi$ is equivalent to the norm $(\|\partial_x\varphi\|^2 + \|\partial_z\varphi\|^2 + ||\varphi||^2)^{1/2}$. We then infer from (4.10) and Lemmas 4.3 and 4.4, that $\hat{u}$ being as in these lemmas, that for every $t > 0$, there exists a constant $K = K^4(t,\omega)$ depending on $t,\omega$ and the data, such that*

$$\sup_{0\leq t'\leq\xi\wedge t} ||\hat{u}||^2 + \int_0^{\xi\wedge t} |\hat{u}|_{(2)}^2 ds \leq K^4 < \infty \quad a.s. \tag{4.38}$$

## 4.3 Strong estimates for U

With the above preliminaries now in hand we may now proceed to study $U$ in the strong norms, the final step of the proof of global existence.

**Lemma 4.5.** *Suppose that $0 < n < \infty$ is a deterministic constant and let $\tau_n \leq \xi$ be the stopping time defined by*

$$\tau_n = \inf\left\{ t \geq 0 : \int_0^{\xi\wedge t} |u|_{(2)}^2 dt' > n \right\} \wedge \xi. \tag{4.39}$$

*Then, for any $t > 0$ there exists a deterministic constant $K = K_n^5(t)$ depending on $n$,$t$ and the data, such that:*

$$\mathbb{E}\left( \sup_{0\leq t'\leq\tau_n\wedge t} \|U\|^2 + \int_0^{\tau_n\wedge t} |AU|^2 dt' \right) \leq K_n^5(t). \tag{4.40}$$

*Proof.* By the Itō formula and truncation argument (see [3]) we derive an equation for $t \mapsto \|U(t)\|$:

$$\begin{aligned} d\|U\|^2 + &2|AU|^2 dt \\ &= (2\langle F - A_pU - B(U) - EU, AU\rangle + \|\sigma(U)\|_{L_2(\mathfrak{U},V)}^2)dt \\ &\quad + 2\langle A^{1/2}\sigma(U), A^{1/2}U\rangle dW. \end{aligned} \tag{4.41}$$

Note that due to Proposition 3.1 this equality holds on the interval $[0,\xi)$.

Fix arbitrary stopping times $0 \leq \tau_a \leq \tau_b \leq \tau_n \wedge t$. We now make estimates of (4.41) on this interval in order to apply the stochastic version of the Gronwall

lemma in [20, Lemma 5.3]. As typical, the stochastic terms are majorized by applying the Burkholder-Davis-Gundy inequality (2.25),

$$\begin{aligned}
\mathbb{E} \sup_{\tau_a \leq t' \leq \tau_b} & \left| \int_{\tau_a}^{t'} \langle A^{1/2}\sigma(U), A^{1/2}U\rangle dW \right| \\
&\leq c\, \mathbb{E} \left( \int_{\tau_a}^{\tau_b} \langle A^{1/2}\sigma(U), A^{1/2}U\rangle^2_{L_2(\mathfrak{U},H)} dt' \right)^{1/2} \\
&\leq \frac{1}{2}\mathbb{E}\left( \sup_{\tau_a \leq t' \leq \tau_b} \|U\|^2 \right) + c\, \mathbb{E} \int_{\tau_a}^{\tau_b} (1 + \|U\|^2) ds.
\end{aligned}$$

By applying (2.14) we may estimate the nonlinear part of the equation

$$|\langle B(U), AU\rangle| \leq c\|u\|^{1/2}|u|^{1/2}_{(2)}\|U\||AU| \leq c|u|^2_{(2)}\|U\|^2 + \frac{1}{4}|AU|^2$$

Making use of these two observations and obvious applications of Young's inequality for the lower order terms (see (2.7), (2.23) ) we may estimate

$$\begin{aligned}
\mathbb{E} & \left( \sup_{\tau_a \leq t' \leq \tau_b} \|U\|^2 + \int_{\tau_a}^{\tau_b} |AU|^2 dt' \right) \\
&\leq c\, \mathbb{E}\|U(\tau_a)\|^2 + c\, \mathbb{E} \int_{\tau_a}^{\tau_b} (1 + |F|^2 + (1 + |u|^2_{(2)})\|U\|^2) dt'.
\end{aligned} \tag{4.42}$$

The Gronwall lemma in [20] applies to real valued, non-negative processes $X, Y, Z, R$ defined on an interval of time $[0, T)$, and such that, for a stopping time $0 < \tau < T$,

$$\mathbb{E} \int_0^\tau (RX + Z) ds < \infty,$$

and such that $\int_0^\tau R ds \leq k$ a.s. Assuming that, for all stopping times $0 \leq \tau_a < \tau_b < \tau$

$$\mathbb{E}( \sup_{\tau_a < t < \tau_b} X + \int_{\tau_a}^{\tau_b} Y ds) \leq C_0 \left( \mathbb{E}(X(\tau_a) + \int_{\tau_a}^{\tau_b} (RX + Z) ds \right)$$

where $C_0$ is a constant independent of the choice of $\tau_a$ and $\tau_b$, then

$$\mathbb{E}\left( \sup_{0 < t < \tau} X + \int_0^\tau Y ds \right) \leq C\mathbb{E}\left( X(0) + \int_0^\tau Z ds \right),$$

where $C = C(C_0, T, K)$. We now just apply this lemma with $\tau = \tau_n, X = ||U||^2, Y = |AU|^2, R = c(1 + |u|^2_{(2)}), Z = c(1 + |F|^2)$ and the result follows. $\square$

## 4.4 Stopping time arguments

We now implement the stopping time arguments that, applied in combination with Lemmas 4.1 - 4.5, imply that $\xi = \infty$.

We define the stochastic processes

$$
\begin{aligned}
X_1(t) &:= \sup_{0\le t'\le t\wedge\xi} |\partial_z \hat{u}|^2 + \int_0^{t\wedge\xi} \|\partial_z \hat{u}\|^2 dt' \\
X_2(t) &:= \sup_{0\le t'\le t\wedge\xi} |\partial_x \hat{u}|^2 + \int_0^{t\wedge\xi} \|\partial_x \hat{u}\|^2 dt' \\
X(t) &:= \sup_{0\le t'\le t\wedge\xi} \|U\|^2 + \int_0^{t\wedge\xi} |AU|^2 dt'
\end{aligned}
\tag{4.43}
$$

and recall, with Lemmas 4.3 and 4.4 that $X_1(t)$ and $X_2(t)$ are almost surely finite for all $t \ge 0$. For $X(t)$, it follows from Lemma 4.5 that $X(t)$ is a.s. finite for every $t \in [0, \tau_n]$ where $\tau_n$ is defined by (4.39).

We first aim to show that $\tau_n \uparrow \infty$ a.s. as $n \to \infty$. Recalling that $u = \hat{u} + \check{u}$, we observe that $|u|^2_{(2)} \le 2|\hat{u}|_{(2)} + 2|\check{u}|^2_{(2)}$ and infer, with Chebyshev's inequality, that for any $t > 0$,

$$
\begin{aligned}
\mathbb{P}(\tau_n < t) &\le \mathbb{P}\left(\int_0^{\xi\wedge t} |u|^2_{(2)} ds > n\right) \\
&\le \mathbb{P}\left(\int_0^{\xi\wedge t} |\hat{u}|^2_{(2)} ds > \frac{n}{2}\right) + \mathbb{P}\left(\int_0^{\xi\wedge t} |\check{u}|^2_{(2)} ds > \frac{n}{2}\right) \\
&\le \mathbb{P}\left(X_1(t) + X_2(t) > cn\right) + \frac{c}{n}\mathbb{E}\int_0^t |\check{u}|^2_{(2)} ds.
\end{aligned}
$$

Thanks to (4.3) this implies that

$$
\lim_{n\to\infty} \mathbb{P}(\tau_n < t) \le \mathbb{P}(X_1(t) + X_2(t) = \infty) = 0.
$$

Observing that the sequence $\tau_n$ is a.s. increasing, we have

$$
\mathbb{P}\left(\lim_{n\to\infty} \tau_n < t\right) = \lim_{n\to\infty} \mathbb{P}(\tau_n < t) = 0,
$$

and hence $\tau_n \uparrow \infty$ a.s. as $n \to \infty$.

We now consider, for any $M > 0$, the stopping time

$$
\sigma_M = \inf\{r \ge 0 : X(r) > M\}
$$

and, in view of applying Proposition 5.1 below we want to evaluate $\mathbb{E}X(\tau_n \wedge \sigma_M \wedge t)$. To this end, we employ Lemma 4.5 and infer that

$$
\sup_M \mathbb{E}X(\tau_n \wedge \sigma_M \wedge t) \le K_n^5(t) < \infty.
$$

We finally conclude, by invoking Proposition 5.1, that $X(t) < \infty$ for any $t > 0$. This implies

$$
X(\xi(\omega)) < \infty \text{ for a.a. } \omega \in \{\xi < \infty\} \tag{4.44}
$$

but since $(U, \xi)$ is a maximal strong solution (cf. (2.35)), we perforce conclude that $\xi = \infty$ a.s. The proof of Theorem 2.1 is thus complete.

# 5 Appendix I: An Abstract Stopping Time Result

We have made use of the following new result in the final steps of the proof above of global existence.

**Proposition 5.1.** *Fix* $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$*, a filtered probability space. Let* $X : \Omega \times [0,\infty) \to \mathbb{R}^+ \cup \{\infty\}$ *be an increasing cádlág stochastic process and define*

$$\sigma_M = \inf\{r \geq 0 : X(r) \geq M\}.$$

*Suppose that there exists an increasing sequence of stopping times* $\tau_n$ *such that* $\tau_n \uparrow \infty$ *a.s. and such that for any fixed* $n > 0$, $t > 0$*:*

$$\kappa_{n,t} := \sup_M \mathbb{E} X(\tau_n \wedge \sigma_M \wedge t) < \infty.$$

*Then, for a set* $\tilde{\Omega} \subset \Omega$ *of full measure,*

$$X(t,\omega) < \infty, \quad \textit{for all } t \in [0,\infty), \omega \in \tilde{\Omega}. \tag{5.1}$$

*Proof.* It is sufficient to show that $\lim_{M\to\infty} \mathbb{P}(\sigma_M < t) = 0$. Indeed since

$$\{X(t) < M\} \subseteq \{\sigma_M \geq t\}$$

and since $\sigma_M$ is an increasing function of $M$, for any $M' > M$,

$$\{\sigma_M \geq t\} \subseteq \{\sigma_{M'} \geq t\},$$

we have that

$$\begin{aligned}
\mathbb{P}(X(t) < \infty) &= \mathbb{P}(\cup_{M>0}\{X(t) < M\}) \\
&\leq \mathbb{P}(\cup_{M>0}\{\sigma_M \geq t\}) \\
&= \lim_{M\to\infty} \mathbb{P}(\sigma_M \geq t) \\
&= \lim_{M\to\infty} (1 - \mathbb{P}(\sigma_M < t)).
\end{aligned}$$

Give any $M, n$, observe that since $X$ is right continuous and increasing,

$$\begin{aligned}
\{\sigma_M < t, \tau_n \geq t\} &= \{X(\sigma_M \wedge t) \geq M, \sigma_M < t, \tau_n \geq t\} \\
&\subseteq \{X(\sigma_M \wedge t) \geq M, \tau_n \geq t\} \\
&\subseteq \{X(\sigma_M \wedge \tau_n \wedge t) \geq M\},
\end{aligned}$$

and therefore

$$\begin{aligned}
\mathbb{P}(\sigma_M < t) &\leq \mathbb{P}(\sigma_M < t, \tau_n \geq t) + \mathbb{P}(\tau_n < t) \\
&\leq \mathbb{P}(X(\sigma_M \wedge \tau_n \wedge t) \geq M) + \mathbb{P}(\tau_n < t) \\
&\leq \frac{1}{M}\mathbb{E}(X(\sigma_M \wedge \tau_n \wedge t)) + \mathbb{P}(\tau_n < t) \\
&\leq \frac{\kappa_{n,t}}{M} + \mathbb{P}(\tau_n < t).
\end{aligned}$$

Thus, for any fixed $n$ and $t$

$$\lim_{M\to\infty} \mathbb{P}(\sigma_M < t) \leq \mathbb{P}(\tau_n < t).$$

However, given the assumptions on $\tau_n$, we have that

$$\lim_{n\to\infty} \mathbb{P}(\tau_n < t) = 0,$$

which shows that $X(t,\omega) < \infty$ a.s. for $\omega \in \Omega$. To determine the set $\tilde{\Omega}$ in 5.1 and complete the proof, we observe that $X$ is an increasing function of $t$ and call, for each $j \in \mathbb{N}, \Omega_j$ the set of full measure such that $X(j,\omega) < \infty, \ \forall \omega \in \Omega_j$. Then $X(t,\omega) < \infty$ for every $t, 0 \leq t \leq j$, and we can take for $\tilde{\Omega}$, the intersection $\cap_{j\geq 1}\Omega_j$ which is a set of full measure as well. □

## Acknowledgments

This work was partially supported by the National Science Foundation under the grants DMS-0604235, DMS-0906440, and DMS- 1004638 and by the Research Fund of Indiana University.

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